Black hole entropy and the holographic principle *

José P. S. Lemos
Centro Multidisciplinar de Astrofísica - CENTRA, Departamento de Física, Instituto Superior Técnico - IST, Universidade Técnica de Lisboa - UTL, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal †

Black holes monopolize nowadays the center stage of fundamental physics. Yet, they are poorly understood objects. Notwithstanding, from their generic properties, one can infer important clues to what a fundamental theory, a theory that includes gravitation and quantum mechanics, should give. Here we review the classical properties of black holes and their associated event horizons, as well as the quantum and thermodynamic properties, such as the temperature, derived from the Hawking radiation, and the entropy. Then, using the black hole properties we discuss a universal bound on the entropy for any object, or for any given region of spacetime, and finally we present the arguments, first given by 't Hooft, that, associating entropy with the number of quantum degrees of freedom, i.e., the logarithm of quantum states, via statistical physics, leads to the conclusion that the degrees of freedom of a given region are in the area A of the region, rather than in its volume V as naïvely could be thought. Surely, a fundamental theory has to take this in consideration.

I. INTRODUCTION

Black holes have been playing a fascinating role in the development of physics. They have entered into the physics domain through a combination of the disciplines of general relativity and astrophysics. Indeed, black holes arise naturally within the theory of general relativity, Einstein's geometric theory of gravitation. Being exact vacuum solutions of the theory, they are thus, unequivocally, geometric objects. From the first solution in 1916, the Schwarzschild black hole, to the rotating Kerr black hole solution found in 1963, until they were accepted as the ultimate endpoints of the gravitational collapse of massive stars, as well as the gravitational collapse of huge amounts of matter (being it, clusters of stars, dark matter or any other matter form) in the center of galaxies, there has been a highly winding road (see, e.g.,[1,2,3] for the initial developments and references therein). The name black hole was coined in 1968 by Wheeler[4] (see also[5,6]). Now there is no doubt that solar mass black holes abound in our Galaxy, and supermassive ones reign at the centers of galaxies playing their roles as energizers of their own neighborhoods, such as in quasars (see[7] for the status of supermassive black holes in galaxies). New theoretical developments show that black holes can form in various ways. They can be eternal being out there since the very initial universe, they can be pair created in a Schwinger type process, they can form from the collision of highly energetic particles as in accelerators, and in the more usually case, discussed in the original work that gave rise to the concept, they can form from the collapse of matter. These and other developments, which have put black holes at the center of studies in fundamental physics, were possible after Hawking discovered that they act as thermal quantum creators and radiators of particles[8]. This result, albeit in a semiclassical regime, unites in one stroke, gravitation (represented by the universal constant of gravitation G, and the velocity of light c), quantum mechanics (represented by Planck's constant $\hbar$) and statistical physics (represented by Boltzmann's constant $k_{\rm B}$). Thus, black holes turn out to be in the forefront of physics, since by acting as unifying objects, through them one can test unifications ideas of gravity (and possibly other fields) and quantum mechanics.

II. CLASSICAL PROPERTIES OF BLACK HOLES

A. The event horizon

General relativity introduces the idea that gravitation is a manifestation of the geometry and curvature of spacetime. Its equations, Einstein's equations, imply that objects, like test particles either massive or massless (like light), move as geodesics in the given underlying curved geometry which, in turn, is established by a certain concentration of matter and energy. Einstein's equations imply in addition that a high concentration of matter and energy curve spacetime strongly. When the concentration of matter and energy is high enough, such as in a collapsing star, spacetime will be

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so curved that it tears itself, so to speak, and forms a black hole. And, like a burst of water in a river, that suddenly opens up a falls, once falling down the falls it is impossible to get back. Indeed, a black hole is a region where the spacetime curvature is so strong that the velocity required to escape from it is greater than the speed of light. The surface of the black hole is the limit of the region from where light cannot escape. Just outside this surface light and particles may escape and be detected at infinity. Inside the surface all particles fall through and never come out again. This boundary, defining what can be seen by observers outside the black hole, the boundary of the region of no return to the outside, is called the event horizon. Thus, in the case the black hole forms from a collapsing star, say, the event horizon is not to be identified with the surface of the star that formed the black hole. The matter that formed the black hole goes through its own horizon, and once inside the horizon, it will continue to collapse right down the falls it created, until a singularity forms, where the curvature of spacetime blows up, i.e., tidal forces disrupt spacetime itself. There is a horizon floating outside, whereas the surface of the star and indeed the whole of the star are now at the singularity. In a black hole, the singularity is hidden behind the horizon, in the hidden region, where nothing comes out. In brief, a black hole is not a solid body with a matter surface, it is a three surface in space and time bounded by a horizon. It is a pure gravitational object with an event horizon, from inside which there is no escape, and with a hidden singularity at its center. A singularity is an object nobody knows what it is. To know what lies inside a black hole, what is a singularity, it is certainly one of the most important problems to be solved in physics, but due to its complexity one sees it seldom discussed. Here we also do not discuss it. Rather, we are interested in the horizon and in the black hole properties, classical and quantum, exterior to it.

In the simplest case the horizon is a sphere, and the corresponding solution is called the Schwarzschild solution. In this case the horizon radius $R_{\rm bh}$ scales with the mass M of the black hole, $R_{\rm bh} = 2M\left(\frac{G}{c^2}\right)$, or in natural or Planck units ($G = 1$, $c = 1$, $\hbar = 1$, $k_{\rm B} = 1$, units which will be often used), one has

$$R_{\rm bh} = 2M\,, \tag{1}$$

so that the more massive the black hole the larger the horizon (see, e.g.,[3]). One can think of the horizon as a sphere of photons, or null geodesics, that are trying to get out radially, but due to the strong gravitational force, i.e., high curvature, stay fixed at $R_{\rm bh} = 2M$. Photon spheres inside the horizon, in the hidden region, are dragged down to the singularity even if they are locally outgoing. On the other hand, photon spheres outside the horizon, moving radially, reach infinity, with those that originate in the horizon's vicinity having to climb a huge gravitational field, or geometrical barrier, take a long time to do so.

B. Black holes have no hair

In general relativity, time depends on the observer. Observers, particles, whatever that enters the black hole will go straight to the singularity, and will not come back, according to their own reckoning. On the other hand, observers that stay outside the horizon, see things differently. They cannot know what happens inside a black hole. Classically, nothing, no signal nor information, can emerge across the horizon and escape into the outside to be detected by an observer in the external world. However, at least one feature of the spherical black hole can be measured outside. One can know from the outside the mass of the black hole. Thus, if the Schwarzschild black hole was formed from the collapse of matter, the only property of the initial matter that can be known after it has collapsed is its mass, all the knowledge on the other properties of the initial matter has disappeared down the horizon. Moreover, additional matter falling into the black hole simply adds to the mass of the black hole, and disappears from sight, taking with itself its own properties. More generally, within general relativity, there are three parameters that can characterize a black hole. A spherically symmetric vacuum black hole, the Schwarzschild black hole, is characterized only by its mass M, with the horizon radius given by equation (1). A great deal of complexity is added if in addition to the mass M the black hole possesses angular momentum J, with the horizon being now oblate instead of spherical. This is the important rotating Kerr black hole solution, an exact solution, that by introducing new dynamics, gave a totally new flare to classical black hole theory, which in turn was essential in the construction of the coupling of black holes to quantum mechanics. Adding electric charge Q one has the Kerr-Newman black hole family, where for $J = 0$ one calls the charged nonrotating black hole a Reissner-Nordström black hole. So, a generic Kerr-Newman black hole is characterized three parameters only, namely (M, J, Q), this being the only knowledge one obtains out of a black hole (see, e.g.,[3]). Such a black hole can form from the collapse of an extremely complex rotating distribution of ions, electrons, radiation, all kinds of other matter, and myriads of other properties characterizing the matter itself. But once it has formed, for an external observer, the only parameters one can know from the outside are the the mass-energy M of the matter that went in, its angular momentum J, and its electrical charge Q. One then says a black hole has no hair, since it has only three hairs, and someone with three hairs is effectively bald (see, e.g.,[3]). This property has measurable consequences on the spacetime outside the horizon. The black hole's mass, angular

momentum, and electric charge certainly change accordingly to the type of matter that is added onto it. Moreover, this change obeys strict rules, indeed, one can show that the laws of conservation of energy, angular momentum and charge, are still valid when a black hole is involved[9]. Thus, M, J, and Q are observable properties that can be known through some form of external interaction with the black hole. On the other hand, all the other features that could possibly further characterize the black hole do not exist. Or if they exist they have vanished from sight. These other features certainly characterize usual matter, star matter say. But once the star has imploded into a black hole all the features, but three, disappear. Where are now those features? Is this property of hiding features, one that one can capitalize on, and discover new properties of the world? By a remarkable twist, quantum mechanics comes to the rescue.

III. QUANTUM PROPERTIES OF BLACK HOLES

The questions raised lead us into the quantum realm and put black holes on a central scene to unify in one stroke, gravitation and quantum mechanics itself, within the framework of thermodynamics and through the concepts of black hole temperature and entropy. These results will also definitely impel into an ultimate bound on the entropy of a given region and to the the establishment of a new revolutionary holographic principle. Let us see each point one at a time, and then all altogether.

There are now many theories, of general relativity type, that have many different black hole solutions, with charges other than M, J, and Q. These theories, one way or the other have general relativity as a limit. So let us stick to general relativity, and moreover let us study the simplest case the Schwarzschild black hole, with its only one hair, the mass M and the associated horizon radius R given in equation (1). The Kerr black hole was very important to put nontrivial dynamics on a classical setting, it acted in this context as a catalyst, but after it induced forcefully the introduction of quantum phenomena in the whole scheme, one can use the simplest black hole, the Schwarzschild black hole, this is indeed sufficient to understand the profound ideas that lie underneath black hole physics.

A. Black hole thermodynamics

The first law of thermodynamics states that the total energy is conserved in an isolated system. It seems trivially obvious nowadays, but some time ago it was hard to understand the nature of heat as energy, an achievement that was accomplished after the work of Carnot first, and the further insights of Mayer, Joule, Kelvin, and Helmholtz in the 1840s[10,11,12]. In its simple form it states that $dE = dQ$, where Q is the amount of heat exchanged, and dE is the variation in internal energy of the system. Another important idea in thermodynamics, and in physics in general, is the introduction of the concept of dQ/T, heat over temperature, a state variable. It was devised by Clausius in 1854, who also found an adequate name for it in 1865, entropy S, such that $dS = dQ/T$. In terms of state variables the first law can now be written as $dE = T\,dS$. This concept of entropy, also led Clausius to postulate a second law of thermodynamics by stating that equilibrium states have an entropy associated with them such that processes can occur only when the final entropy is larger than the initial entropy, i.e., in any closed system, entropy always increases or stays the same, $dS \geq 0$. As also worked out by him in a paper dealing with the "the nature of the motion which we call heat", the entropy concept had an immediate impact in kinetic theory and statistical mechanics (see, e.g.,[12]). Both these advances are remarkable. Of course, Clausius could not know of its ramifications and problems that such a concept would introduce more than 100 years later, when applied to gravitating systems. A first hint of these problems appeared in studies on Newtonian gravitating systems, such as in clusters of stars[13,14], whereas one needs general relativity to apply thermodynamic and statistical mechanics concepts to fully general relativistic objects. Two such systems are the Universe itself and a black hole. That the entropy concept and its associated second law of entropy increase can have remarkable implications upon the Universe as a whole and on the arrow of time was first understood by Boltzmann within his statistical physics formulation (see, e.g.,[12,15]), an issue that is still today under heavy discussion[16,17]. But conundrums of a different caliber and with a more direct physical significance perhaps, involving physics at the most fundamental level, have arisen from the fact that a black hole has entropy, an entropy with a form never seen before. Indeed, black hole entropy is proportional to the area of the black hole, rather than the volume. Let us first see how the black hole entropy arises and then where the second law takes us to.

If one thinks, as before, of a black hole forming from the collapse of a matter star, one has an initial configuration, a star say, and a final configuration, a black hole. The star is specified by very many parameters and quantities, the black hole by the mass M alone, in the spherical vacuum case. This led us to argue above that a black hole is a system specified by one macroscopic hair parameter only, the mass M and hiding lots of other parameters perhaps located inside the black hole event horizon. Thus, the black hole acts like a black box. In physics there is another instance of this kind of black box situation, whereby a system is specified and usefully described by few parameters, but on

a closer look there are many more other parameters that are not accounted for in the gross macroscopic description. This is the well know case of thermodynamics described above. For thermodynamical systems in equilibrium one gives the energy E, the volume V, and the number of particles N, say, and one can describe the system in a useful manner, obtaining from the laws of the thermodynamics its entropy and other important quantities. On doing this one does not worry that the system encloses a huge number of molecules and that the description hides its own microscopic features. Of course, one can then plunge into a deeper treatment and apply statistical mechanical methods to the particles constituting the thermodynamical system, using the distribution density function of Gibbs for classical particles or the density matrix for quantum ones in the appropriate ensemble, and then applying Boltzmann's formula for the entropy $S = k_{\rm B} \ln\Omega$, where Ω is the number of states, or any other formula, like Gibbs' formula, to make the connection to thermodynamics. Due to this black box analogy between a black hole and a thermodynamic system, one can ask first the question: Is thus a black hole a thermodynamic system? If yes, one should pursue and ask two further questions: Can one find the analogue of the constituent particles to allow for a statistical interpretation? To where can the black hole thermodynamic system lead us to, in terms of the ultimate fundamental theory?

The first, and then the subsequent questions, started to be answered through a combination of hints. From the Penrose and superradiance processes, deduced using Kerr black hole backgrounds, one could conclude that the area of a black hole would not decrease in such cases[18,19,20,21], an idea that culminated with the underlying area law theorem, which states that in a broad class of circumstances, such as in black hole merger events, the area could never decrease, only increase or stay even in any process[22]. At about the same time, Wheeler raised the problem (see[5]), that when matter disappears into a black hole, its entropy is gone for good, and the second law seems to be transcended, i.e., in the vicinity of a black hole entropy can be dumped onto it, thus disappearing from the outside world, and grossly violating the second law of thermodynamics. Bekenstein, a Ph.D. student in Princeton at the time, solved part of the problem in one stroke. With the hint that the black hole area always increases, he postulated, entropy is area[23]. Specifically, he postulated[23], $S_{\rm bh} = \eta \,\frac{A_{\rm bh}}{A_{\rm pl}}\; k_{\rm B}$, where $A_{\rm bh}$ is the black hole area, η is a number of the order of unity or so, that could not be determined, $A_{\rm pl}$ is the Planck area, and $k_{\rm B}$ is the Boltzmann constant. Note that the Planck length $l_{\rm pl} \equiv \sqrt{\frac{G\hbar}{c^2}}$, of the order of 10^{-33} cm, is the fundamental length scale related to gravity and quantum mechanics, and the Planck area is its square, $A_{\rm pl} = l_{\rm pl}^2 \sim 10^{-66}\,{\rm cm}^2$. Several physical arguments were invoked to why the entropy S should go with $A_{\rm bh}$ and not with $\sqrt{A_{\rm bh}}$ or $A_{\rm bh}^2$. For instance, it cannot go with $\sqrt{A_{\rm bh}}$. This is because $A_{\rm bh}$ itself goes with $R_{\rm bh}^2 \sim M^2$, for a Schwarzschild black hole, and when two black holes of masses M_1 and M_2 merge, the final mass M obeys $M < M_1 + M_2$ since there is emission of gravitational radiation. But if $S_{\rm bh} \propto \sqrt{A_{\rm bh}} \propto M < M_1 + M_2 \propto S_{\rm bh1} + S_{\rm bh2}$ the entropy could decrease, so such a law is no good. The correct option turns out to be $S_{\rm bh} \propto A_{\rm bh}$, the one that Bekenstein took. It seems thus, there is indeed a link between black holes and thermodynamics. In addition, it seems correct to understand that this phenomenum is a manifestation of an underlying fundamental theory of spacetime, a quantum theory of gravity, since the Planck area appears naturally in the formula, hinting that there must be a connection with some fundamental spacetime microscopic ingredient whose statistics connects to the thermodynamics.

Since there is a link between black holes and thermodynamics, black holes have entropy, one can then wonder whether black holes obey the first and second laws of thermodynamics (see, e.g.,[24,25,26] for reviews on black hole particle creation and black hole thermodynamics). In relation to the first law, note that for a Schwarzschild black hole, the simplest case, one has that the area of the event horizon is given precisely by $A_{\rm bh} = 4\pi\, R_{\rm bh}^2$. Now, $R_{\rm bh} = 2M$, so one has $A_{\rm bh} = 16\pi M^2$ (in natural or Planck units). Then one finds $dM = 1/(32\,\pi\, M)\, dA_{\rm bh}$, which can be written as[9],

$$dM = \frac{\kappa}{8\,\pi}\, dA_{\rm bh}\,, \tag{2}$$

where κ is the surface gravity of the black hole, a quantity that can be calculated independently and gives a measure of the acceleration of a particle at the event horizon. In the Schwarzschild case $\kappa = 1/4\,M$. Equation (2) is a simple dynamical equation for the black hole. When one compares it with the first law of thermodynamics, $dE = TdS$, the similarity is striking, and since following Bekenstein $S_{\rm bh}$ and $A_{\rm bh}$ are linked, and following Einstein M and E are linked, indeed they are the same quantity, one is tempted to associate T and κ[27]. But from thermodynamical arguments alone one cannot determine η the dimensionless proportionality constant of order unity between entropy and area, and cannot also determine the constant of proportionality between T and κ, related to η. Using quantum field theory methods in curved spacetime Hawking[8], in a spectacular tour de force, showed that a Schwarzschild black hole radiates quantically as a black body at temperature $T_{\rm bh} = \frac{1}{8\pi M}\left(\frac{\hbar\, c^3}{G\, k_{\rm B}}\right)$, uniting in one formula $\hbar$, G and c, and $k_{\rm B}$. In natural Planck units, and returning to κ this is,

$$T_{\rm bh} = \frac{\kappa}{2\pi}\,, \tag{3}$$

connecting definitely and physically the surface gravity with temperature, and closing the thermodynamic link. Moreover, from the first law of thermodynamics one obtains $\eta = 1/4$, yielding finally

$$S_{\rm bh} = \frac{1}{4}\,A_{\rm bh}\,, \tag{4}$$

in natural units. Thus, Hawking radiation allows one to determine, on one hand, the relation between the temperature of the black hole and its surface gravity, and on the other hand, to fix once and for all the proportionality constant between black hole entropy and horizon area. The black hole entropy is one quarter of the event horizon's area, when measured in Planck area units. For thermodynamic systems, this is a huge entropy, the entropy of a black hole one centimeter in radius is about 10^{66} in Planck units, of the order of the thermodynamic entropy of a cloud of water with 10^{-3} light years in radius. The Hawking radiation solved definitely the thermodynamic conundrum. The generalized first law is then given by a simple extension of equation (2). M is now the energy of the whole system, black hole plus matter, T for the matter and $\kappa/2\pi$ for the black hole have the same values, for a system in equilibrium, and the entropy of the thermodynamic system is now $S = S_{\rm bh} + S_{\rm matter}$, a sum of the black hole entropy $S_{\rm bh}$, and the usual entropy of the matter and radiation fields which we denote simply as $S_{\rm matter}$. It is advisable to separate the entropy into two terms, since one does not know for sure the meaning of black hole entropy.

What about the second law of thermodynamics, can it be embodied in a framework where black holes are present? The second law of thermodynamics mathematizes the evidence that many processes in nature are irreversible, hot coffee cools in the atmosphere, but cold coffee never gets hot spontaneously, and so on. The law states that the entropy of an isolated physical system never decreases, either remains constant, or it increases, usually. It holds in a world where gravitational physics is unimportant. What happens in gravitational systems in which there are black holes. Given that the black hole is a thermodynamic system, with entropy and temperature well defined, the second law of thermodynamics $d\,S \geq 0$ should be obeyed. Indeed one can write the second law as

$$dS_{\rm bh} + dS_{\rm matter} \geq 0\,, \tag{5}$$

commonly called the generalized second law[28]. In words, the sum of the black hole entropy and the ordinary entropy outside the black hole cannot decrease. This generalized second law proved important in many developments, and its consequences are the main object of this review. The generalized second law has passed several tests. For instance, when a star collapses to form a black hole, one can show that the black hole has an entropy that far exceeds the initial entropy of the star. Also, when matter falls into an already existing black hole, the increase in black hole entropy always compensates for the lost entropy of the matter down the horizon. Another interesting example where the generalized second law holds involves Hawking radiation. Due to this radiation the black hole evanesces. Its mass decreases, and so the black hole area also decreases. This violates the area law theorem, but this is no problem, the theorem was proved classically. Then, the black hole entropy decreases indeed. However, one can show that the entropy in the emitted radiation exceeds by some amount the original entropy of the black hole, upholding the generalized second law[29]. Using generic arguments hinged on a quantum definition of entropy, it is possible to argue, that due to lack of influence of the inside on the outside, the generalized second law is valid for processes involving black holes[30]. We note that there are arguments that claim that one does not need the generalized second law, the ordinary second law alone is enough in itself, see, e.g.,[31,32,33,34]. This controversy would merit a review in itself.

B. Black hole entropy

Before start discussing to where the generalized second law leads us, it is interesting to think about the consequences of black holes having entropy, as Bekenstein did almost immediately after his major discovery[23]. Entropy is one of the most important concepts in everyday physics. Somehow, it is a recondite concept, and even more mysterious when black holes are involved. Let us see this.

Following Boltzmann, the entropy S of a closed isolated system with fixed macroscopic parameters, is given by,

$$S = k_{\rm B} \ln \Omega\,, \tag{6}$$

where again $k_{\rm B}$ is the Boltzmann constant, and Ω is the number of accessible microstates that the large system has. Each microstate i has equal probability p_i of occurring, so $p_i = 1/\Omega$, and equation (6) can be written in the alternative form $S = -k_{\rm B} \ln\, p_i$. For open systems, that can exchange energy and other quantities, the entropy can be written in a more useful manner as $S = -k_{\rm B} \sum_i p_i \ln p_i$, where p_i is the probability of microstate i occurring, which now due to the openness of the system is not anymore equal for each state, states with a given energy, the average energy, have a higher probability of occurring. This formula was given by Gibbs upon careful consideration of his

ensemble theory and generalization of Boltzmann ideas (see, e.g.,[35] for the deduction of Gibbs entropy formula from equation (6)). If the system is closed then $p_i = 1/\Omega$ and Boltzmann equation (6) follows. In the Gibbs formulation, one works in a $6N$ dimensional classical phase space, and having to work with a continuum distribution probability density, the phase-space density ρ (instead of p_i), one should write $S = -k_{\rm B} \int d^{3N}q\, d^{3N}p\ \rho(q,p)\, \ln \rho(q,p)$, which is the continuum Gibbs entropy equation for a system of N particles in three dimensional space with $6N$ classical degrees of freedom, $3N$ for the coordinates and $3N$ for the momenta. In this setting each point in the phase-space represents a state, a microstate, of the system. This was then generalized, in a natural way, although through a postulated basis, by Von Neumann to quantum systems. One postulates first that ρ goes into the quantum operator $\hat\rho$ which gives the probability that the system is in some given microstate (essentially is p_i), and second that the entropy is $S = -k_{\rm B}{\rm Tr}\, \hat\rho \ln \hat\rho$. This Von Neumann entropy should be calculated in some complete orthonormal basis of the appropriate state space, or Hilbert space. Since $\hat\rho$ can have non-diagonal terms, which can be suited for calculating quantities other than traces, the Von Neumann entropy is a generalization of the Gibbs entropy, although possibly not unique. All these formulas for the entropy can be useful, depending on the context one is working. Gibbs formula, for instance, has an interesting advantage sometimes. Indeed, the formula is the same as the one that emerged for the entropy in information theory, the Shannon entropy[36]. The Shannon entropy first appeared in connection with a mathematical theory of communication, where it was perceived that the best measure of information is entropy. In fact, entropy in an informational context represents missing information. The Shannon entropy formula is given by $S = -k_{\rm S} \sum_i p_i \ln p_i$, where here, since the connection with temperature is unimportant, $k_{\rm B}$ is substituted by $k_{\rm S}$, the Shannon constant, which generally is put equal to $1/\ln 2$, so that the entropy is given in bits, a dimensionless quantity. Apart the constant used, $k_{\rm B}$ or $k_{\rm S}$, which is a matter of convenience, the two entropies are the same. However, Shannon entropy is applied to measure the information a given system (a computer for instance) has, basically how many bits the system has, whereas Gibbs entropy is applied to the thermodynamic system itself, essentially the number of molecules the system (a computer for instance) has. Both entropies can be given in Shannon units, of course. Gibbs entropy is usually much larger than Shannon entropy. The day bits are imprinted on molecules, rather than in chips, the two entropies will give the same number. The connection between information and entropy turns out to be very useful and important in black hole theory, see, e.g.,[37].

To try to understand the meaning of the black hole entropy given in equation (4), one can use the various formulas for the entropy presented above. But here, for our purposes, it is simpler if we explore Boltzmann's formula (6). It chiefly claims that one way to think about entropy is that it is a measure, a logarithmic measure, of the number of accessible microstates that the isolated system has. Any system, including a black hole, should follow this rule. For black holes, there is a snag, we do not really know what those microstates are, so we cannot count them to take the entropy. There are several ideas. One idea is that the microstates could be associated to the singularity inside the event horizon, where the crushed matter and the demolished spacetime lie altogether. As in the ordinary matter case, one could think that rearranging these states, somehow lying on the singularity, do not affect the mass M of the black hole (and Q, and J for the other hairs, if there are those). There are problems with this interpretation for the entropy. The singularity is in principle spacelike, in addition it is certainly causally disconnected to the outside of the black hole, and therefore it is hard to imagine how it could influence any quantity exterior to itself, let alone to the exterior of event horizon. This interpretation is related to the interpretation that the degrees of freedom, are in some measure of the volume inside the horizon (see, e.g.,[38,39]). Moreover, such type of interpretations are very difficult to implement since no one knows really what goes on inside let alone in a a singularity, only with a fully developed theory of quantum gravity can one attempt to understand singularities. Another place where the microstates might be located is in the vicinity of the event horizon area as has been suggested many times (see, e.g.,[40] for a heuristic account,[41,42,43] for a particular implementation, and[44] for a review). The idea beyond this suggestion is that for photons emitted near the horizon, only those with very high energies, indeed trans-Planckian energies, can arrive with some finite nonzero energy at infinity, and so, these photons probe near-horizon Planckian structures, i.e., probe quantum gravity. Indeed, light sent from the very vicinity of the horizon has to climb up the huge gravitational field, or if one prefers, the huge spacetime barrier set up by the black hole. In turn this means that the pulse of light an observer a distance away form the horizon receives has a much lower frequency (much higher wavelength) than the very high pulse frequency (very low wavelength) of the emitted pulse. This is the redshift effect. The nearer the horizon the pulse is emitted the higher the effect. Since in quantum mechanics frequency and energy are the same thing, $E = \hbar\omega$, the closer to the horizon the photon is emitted, the more energy it must get rid off as it travels towards the observer. In effect, there is an exponential gravitational redshift near the horizon so that the outgoing photons and other Hawking radiation particles originate from modes with extremely large, trans-Planckian, energies. But now this is very important, photons with very high energy, very low wavelengths, probe very small regions. So the Planckian and trans-Planckian photons, that arrive at the observer somehow come from regions of space and time that are themselves quantum gravity regions. There are various possibilities for these regions, such spacetime regions may be discrete, or may be fluctuating in a quantum foam structure, or whatever. Thus, if one can observe Hawking photons originating from very close to the horizon of a black hole, one is possibly seeing the quantum structure of the

spacetime. In the context we are discussing, this means that the entropy should be a feature of the horizon region itself. Near the horizon quantum gravity and matter fields are being probed, and these, alone or together, can be the degrees of freedom one is looking for to generate the entropy of the black hole. This fact led thus to the proposal that the entropy is in the horizon area. This proposal is very interesting and may solve the degrees of freedom, or the entropy, problem. But this follow up from black hole thermodynamics is not our main concern here, see[44] and references therein for more on that. We have commented on it solely to get a preliminary understanding of black hole entropy. Even without understanding where are those degrees of freedom that make up the black hole entropy one can derive some new consequences, such as the entropy bounds and the holographic principle.

IV. AN ENTROPY BOUND INVOLVING BLACK HOLES

The generalized second law allows us to set bounds on the the entropy of a given system. Or, in terms of information, it sets bound on the information capacity any isolated physical system can have. Since this law involves gravitation, and gravitation together with quantum mechanics should provide a fundamental theory, the bound refers to the maximum entropy up to the ultimate level of description, a given region can have.

To obtain the bound let us think of the formation of a black hole from the collapse of some ordinary matter. It is interesting to consider thus an initial configuration, a star say, and a final configuration, a black hole. Consider then any approximately spherical isolated matter that is not itself a black hole, and that fits inside a closed surface of area A. If the mass can collapse to a black hole, the black hole will end up with a horizon area smaller than A, i.e., $A_{\rm bh} \leq A$. The black hole entropy, $S_{\rm bh} = A_{\rm bh}/4$, is therefore smaller than $A/4$. According to the generalized second law, the entropy of the system cannot decrease. Therefore, the initial entropy of the matter system, $S^{\rm system}_{\rm initial}$, cannot be larger than $A_{\rm bh}/4$, and so not larger than $A/4$. It follows that the entropy of an isolated physical system with boundary area A is necessary less than $A/4$, i.e., $S^{\rm system}_{\rm initial} < A/4$. So, following ideas devised early by Bekenstein[45], Susskind[46] through such a simple argument developed this spherical bound. Putting $S^{\rm system}_{\rm initial} \equiv S$, to clarify the notation, Susskind's bound reads

$$S \leq \frac{1}{4} A \,. \tag{7}$$

One can now anticipate a result which will be further discussed in the next section: since A is the number of Planck unit areas that tile the area A, the bound says that the number of quantum degrees of freedom, or the logarithm of the number of quantum states, of the system within an area A is necessarily equal or less than one quarter of the number of Planck unit areas that fit in the area A. In brief, following[46] the generalized second law implies the bound (7), usually called the spherical holographic bound for reasons we will see below.

One example one can give that certainly satisfies the bound refers to two black holes in a box. Let us put two Schwarzschild black holes of masses M_1 and M_2 in a box. The entropy is $S = \frac{1}{4}\left(A_{\rm bh1} + A_{\rm bh2}\right)$. Since $A_{\rm bh1} = 4\pi R^2_{\rm bh1} = 16\pi M_1^2$ and $A_{\rm bh2} = 4\pi R^2_{\rm bh2} = 16\pi M_2^2$, one has $S = 4\pi\left(M_1^2 + M_2^2\right)$. Now, from a distance, the system should not be a large black hole of mass $M = M_1 + M_2$, otherwise the argument is of no interest. So, there is a radius for the box R, with the associated area A, which obeys $A_{\rm bh1} + A_{\rm bh} < A$, i.e., $\frac{1}{4}\left(A_{\rm bh1} + A_{\rm bh2}\right) < \frac{1}{4}A$. Finally, since $S = \frac{1}{4}\left(A_{\rm bh1} + A_{\rm bh1}\right)$ one has $S < \frac{1}{4}A$. The bound is clearly satisfied, and it is saturated only when the box is a black hole. There are many other examples one can think of.

Now this bound suffers from some drawbacks, it only applies to systems which are initially nearly spherically symmetric, and not much strongly time dependent. In addition is not a covariant bound, and in general relativity, all statements should some way or another be put in a covariant form. These problems were cured by Bousso[47,48,49], who has managed to formulate a covariant entropy bound, (see also[50,51,52,53]). Susskind's bound is a particular case of this Bousso's covariant bound. In addition, the original Bekenstein bound[45] can be derived through Susskind bound, and surely, through Bousso's bound (see[54]). The covariant entropy bound is fascinating and has been proved correct in very many instances. However, due to its simplicity, it is useful to stick to the spherical bound given in equation (7). This will take us more directly to the holographic principle.

V. HOLOGRAPHIC PRINCIPLE

A. Definition

What are the ultimate degrees of freedom, what are the degrees of freedom of quantum gravity, what are the fundamental constituents of spacetime? Portions of ordinary matter are made of molecules, which are made of atoms,

which are made of electrons and nuclei, which nuclei are made of protons and neutrons, which are made of quarks, and so on including all known interactions and their associated particles, up to the quantum gravity level, the fundamental spacetime level. For ordinary matter and the corresponding cascade of constituents and interactions we know what and where the degrees of freedom are. For spacetime we do not, yet, unfortunately. However, even without knowing of what the spacetime is made of we can extract limits for the number of such degrees of freedom, and other relevant information from the entropy bounds discussed previously.

Indeed, based on his own ideas about entropy bounds and even before the spherical bound was advanced, 't Hooft[55] proposed that the degrees of freedom of a region of space circumscribed within an area A are in the area itself. This is counter to the results of everyday physics which give that the entropy is proportional to the volume of the region, and so the degrees of freedom of these usual systems are in this sense in the volume. A usual system has entropy, or information, inside it. For instance, in a book, the information is contained inside (in the volume), not in the cover (in the area). One knows that reading the title of a book is not enough at all to know what is inside. This also happens for all usual thermodynamic physical systems upon a statistical physics treatment. However, there are system that do not follow this rule. These are the black holes, which once more reveal themselves as the most fundamental objects to uncover the secrets of nature. For black holes, the entropy and information of what is inside is projected in the area. Since black holes are of fundamental importance, both in gravitation and quantum theory, they are the ones that dictate the ultimate rule that should be obeyed. Thus, the generalized second law, derived from black hole physics, together with the bounds above conjure to give the result that the degrees of freedom are in the area itself.

To be definitive, define first the number of degrees of freedom $N_{\rm f}$ of a quantum system as the logarithm of the number of quantum states Ω of the system, with Ω being the same as the dimension of the Hilbert space of the system, (parts of this exposition follows[46,49,55]). So $N_{\rm f} = \ln \Omega$. This generalizes the idea of degrees of freedom of a classical system. To have an idea of what $N_{\rm f}$ means, take, for example, a spin system with 1000 spins, each spin being able to be up or down only. Such a system should have around 1000 degrees of freedom. In fact, from the definition above, since there are two states for each spin, one has that the number of states for the whole system is $\mathcal{N} = 2^{1000}$. So its number of degrees of freedom is $N_{\rm f} = 1000 \ln 2$. In terms of information, following Shannon, this means that the system can store 1000 bits, or its Shannon entropy is 1000, as mentioned in Section III. This system is small, thermodynamics systems are huge in comparison. For a given isolated thermodynamic system, with entropy S, the number of independent quantum states is $\Omega = {\rm e}^S$, in natural units, see equation (6). So, for a thermodynamic system, $N_{\rm f}$ is related to the entropy S, in fact, following the definition above, they are the same in natural units, $S = N_{\rm f}$. For instance, in order to see that such a definition is reasonable, recall that the number of states Ω of an ideal gas with fixed energy E, volume V, and number of particles N can be written as $\Omega = \left[{\rm e}^{5/2}\,(V/N)(4\pi m E/3N)^{3/2}/h^3\right]^N$, so that since $V = L^3$, where L is the dimension of the enclosure say, and $(2mE)^{1/2} = \bar{p}$, where $\bar{p}$ is a typical momentum of the particles, one finds $N_{\rm f} = 6N\,\alpha$ where α is a number of the order one or so, proportional to a logarithm term. Thus, $N_{\rm f}$ as defined gives roughly the classical number $6N$ of degrees of freedom as expected. Of course, through this definition one has exactly $S = N_{\rm f} = 6N\left[5/12 + (1/6)\ln\left((V/N)(4\pi m E/3N)^{3/2}/h^3\right)\right]$, which is the Sackur-Tetrode formula (see, e.g.,[35]).

Let us suppose then that we are given a nearly spherical finite region of volume V with boundary area A. Suppose again that, initially, gravitation is not strong enough, so that spacetime is not time dependent and all the relevant physical quantities are well defined. One can consider then that the nearly spherical region has some matter content. However, this content ultimately does not interest us, we can forget about the solid, liquid, gas, or vacuum that fills up the region. At the ultimate level one is only interested in the region itself, in the spacetime itself alone. One wants to know what are the states themselves of that region and what is their number, at the most fundamental level. So we want to know how many degrees of freedom are there for the fundamental system, or how much complexity there is at the fundamental level, or how much information one needs to specify the region. One way to start out and see where it leads to is to pick up a theory that has given fruitful results in ultra microscopic physics. This theory is quantum field theory. It works extremely well in flat spacetime, and with care it can be extrapolated to curved spacetime[56]. A quantum field is described by harmonic oscillators at every point in spacetime. A quantum harmonic oscillator has an infinite number of states and so an infinite number of quantum degrees of freedom. So, there are infinite number of degrees of freedom at every spacetime point in a quantum field. Moreover, within a volume V there are infinite number of points. Thus, a quantum field in a given spacetime background has, by this rationale, a huge infinite of infinite number of degrees of freedom. So it seems. However, one can easily argue, that the number is indeed huge, albeit finite. Indeed, gravity together with quantum theory show that there is a minimum length scale given by the Planck length $l_{\rm pl}$, and a maximum energy scale, the Planck mass $m_{\rm pl}$, beyond which any theory of distances and scales does not make sense. So, crudely, one might guess that there is one oscillator per $l_{\rm pl}$, each with maximum energy $m_{\rm pl}$ (more energy than this turns spacetime into a black hole). One can now think that each spacetime volume V has $V/V_{\rm pl}$ oscillators and each oscillator has a finite number of states n say, which is large, (the highest energy state for each oscillator is given by the Planck energy). So, in Planck or natural units, the total number of states is $\Omega \simeq n^V$ and thus the number of degrees of freedom is $N_{\rm f} \simeq V \ln n$, i.e., $S \simeq V \ln n$ (see[49] for more details). So, if this

conclusion is fully correct, a fundamental theory needs to account for an entropy proportional to the volume or bulk of each region being considered, i.e., the disorder of the region goes with the volume.

But this naïve reasoning fails when gravity is included. The fundamental theory has to include gravity, for sure, and when this is done one finds that a fundamental theory needs only to account for an entropy proportional to the surface area, and this is much less than entropy proportional to the volume. Let us see this in more detail. Entropy is a measure of the logarithm of the number of microstates of a given system macroscopically specified, so that, as seen, entropy is also a measure of the number of degrees of freedom of the system. Now we know that given an area A there is a bound for the entropy $S \leq \frac{1}{4}A$ in Planck units (i.e., $\frac{S}{k_{\rm B}} \leq \frac{1}{4}\frac{A}{A_{\rm pl}}$ restoring units) for any system. Any system, including the fundamental system, has to obey this bound. When we have an adequate quantum gravity it will give an entropy for the quantum system which is equal or lower than this bound. Now a black hole with this same area A saturates the bound, so one can say there are systems that saturate the bound. Thus the number of degrees of freedom of a sphere of area A, and the related number of states are given by, respectively,

$$N_{\rm f} = \frac{1}{4}A\,, \quad \text{and} \quad \Omega = {\rm e}^{\frac{1}{4}A}\,. \tag{8}$$

That the number of states has to be given by (8) can be argued more effectively using unitarity, which claims that an initial state evolves in a well defined manner to a final state, such that probability in quantum theory is conserved. Essentially, it says one can derive the final state from the initial and vice-versa. Given an initial object, or region, suppose that the number of states of the Hilbert space for it goes roughly with ${\rm e}^V$. Allow the object, or the region, to evolve into a black hole of the same size of the region. Then the new number of states is ${\rm e}^{A/4}$, where A is the area enclosing V. But this number is much less than the initial one, so one cannot recover the initial state from the final one, the states would not evolve unitarily. Thus, one should start with ${\rm e}^{A/4}$ as the initial number of states.

Now, the number given in (8) is much smaller than the number n^V guessed earlier, for lengths larger than about the Planck size. One can understand this much lower number than the one given by the naïve guess of quantum field theory, by invoking heuristic arguments coming from the inclusion of gravity (see again[49] for more details). It is true we have imposed, naïvely, that there is at most one Planck mass per Planck volume. So there is a high energy cut off, and modes with higher energy than that do not exist and do not contribute to the entropy. That is fine. But this cut off at large scales, scales larger than Planck scales, gives that, within a region of radius R and assuming roughly a constant field density, the mass can scale as $M/M_{\rm Pl} \sim (R/R_{\rm Pl})^3$, i.e., $M \sim R^3$ in Planck units. This cannot be right, since we know that for sure $M/M_{\rm Pl} \lesssim R/R_{\rm Pl}$, i.e., $M \lesssim R$ in Planck units. For $M \gtrsim R$ one forms a black hole, the most massive object that can be localized in the sphere of radius R. Thus at face value it seems that one should rather assume that the field content (gravity and possibly other fields) density goes at most as $1/R^2$ rather than constant. And so there are many less states than naïvely one could guess. Due to gravity, a long range universal field, the energy of the field content is lower in large volumes than it could possibly be in small Planckian volumes. Thus crudely, the entropy, which in many ways is related to the energy, is also drastically reduced. The conclusion is that naïve field theory seems to yield more degrees of freedom than those that can be used for generating entropy, or to store information. So, there are at most $A/4$ degrees of freedom inside a region whose volume is surrounded by an area A. Most systems have less than $A/4$ degrees of freedom such as any system made of ordinary matter. One system that strictly matches the bound is a black hole, which has precisely $A/4$ degrees of freedom. If one has any system, and wants to excite more degrees of freedom than those given by the bound, then one forms a black hole. A black hole is an object that has the maximum entropy for an outside observer. Perhaps there can be other objects with such an entropy, e.g., quasi black holes (see[57,58]), but not object has larger entropy. Summing up what we have seen so far, we can say that Bekenstein's and Hawking's works, coupled to the Susskind bound, states that a fundamental theory, one in which gravity is included, has a number of degrees of freedom proportional to the area, which leads to fewer degrees of freedom, and so less entropy or less disorder, than the theory would have to have had the entropy of a region been proportional to the volume, rather than the area.

Then one can go a step further, as 't Hooft did[55], actually before the spherical and the covariant bounds were discussed. If the maximum entropy, obtained from fundamental degrees of freedom, in a given region of space, is proportional to the area, rather than the volume, then the degrees of freedom should lie in the area of the region. This is the basis of the holographic principle. It states: a region with boundary area A is totally described by at most $A/4$ degrees of freedom (in Planck or natural units), i.e., about one bit of information per Planck area. In a sense, the description of the processes that happen within the region's volume, is projected into the surface of that region, in the same way as the visual perception of a three dimensional region can be encoded in a hologram, a two dimensional sheet. So, in principle, there are two possible descriptions, the volumetric or three dimensional, and the areal or two dimensional description, the latter one being certainly more economical. In order to grasp better this idea let us introduce the following allegory[59]. Imagine that a futuristic plane is surrounded by a hypothetical giant two dimensional spherical screen located in space. And that all the activities and happenings on such a planet,

through illumination, are projected onto this screen. The image on the giant screen would be a blow down of the three dimensional world to two dimensions. If the projection is accurate enough, there are two sorts of people, one two dimensional, the other three dimensional. But in such an accurate case, people in both scenarios can think of themselves as equally alive, and the other as the mirage, each containing the same amount of information and each being described through equivalent mathematical theories, with no theory being more correct than the other.

B. Implementation

Concrete examples which satisfy the holographic principle have been found in anti-de Sitter spacetimes, i.e., spacetimes with a negative cosmological constant. A de Sitter universe is one with a positive cosmological constant that creates a universe uniformly accelerating, and present observational results indicate we leave in such a universe. On the other hand, a negative cosmological constant has the property of giving a uniform gravitational attraction over all of the spacetime. Such a spacetime has uniform negative curvature, and due to the relentless constant attraction this spacetime has a boundary, it is as if the spacetime is set up in a box with some definite length. For instance, in this spacetime, a massless particle can travel in a finite time from any point in the interior to spatial infinity and back again.

Anti-de Sitter spacetimes appear often in string theory, or its M-theory generalization, as well as in several corresponding low energy limits that yield supergravity theories. Now, when string theory is properly used in an anti-de Sitter spacetime one finds that it is equivalent to a quantum field theory on the boundary of that spacetime[60]. The first instance in which this holographic result was found is not directly applicable to our real universe, first because the cosmological constant of the universe is positive rather than negative, and second because in[60] it was found that the calculations simplify if one works in a five dimensional (four space and one time dimensions) anti-de Sitter spacetime, AdS_5, rather than the more usual four dimensional one, AdS_4. String theory is well formulated only in ten dimensions, so to be precise, one can also include compactified dimensions. In fact, the example is given in the context of AdS_5 times the five sphere S_5, so that the whole spacetime is $\mathrm{AdS}_5 \times \mathrm{S}_5$. The equivalent boundary quantum field theory arises from the boundary of AdS_5. In this setting, one can argue that the physics experimented by an observer living in the bulk of the AdS_5 spacetime can be completely described in terms of the physics taking place on the spacetime's boundary. This initial result, valid for a five dimensional anti-de Sitter spacetime and its four dimensional boundary dual, was later exhibited in many other situations and other dimensions, including the more usual four dimensional anti-de Sitter spacetime, being in this case dual to a quantum field in three spacetime dimensions. Generically, one finds that the bulk and the boundary descriptions are equivalent, none of the descriptions is more complete or important than the other. In the bulk description gravitation operates and spacetime is d dimensional, say, whereas in the boundary description there is no gravity but a quantum field theory, conformal in nature, operating in a d -1 dimensional flat spacetime. It is as if there is a duality between this d dimensional spacetime and its d -1 dimensional boundary. It then means that two different theories, acting in spacetimes with different dimensions, are equivalent. This reinforces the idea that beings living in the d dimensional spacetime would be mathematically equivalent to beings living in the d -1 dimensional one, there is no way to distinguish between them. This is certainly an interesting implementation of the holographic principle of 't Hooft[55]. Technically, it is also fruitful, since difficult calculations performed on the bulk spacetime can perhaps be easily done in the quantum field theory on the boundary and vice versa. For instance, one can show that a black hole in anti-de Sitter spacetime is equivalent to hot radiation in the boundary, and that the mysterious black hole entropy is equivalent to the radiation entropy[61]. In addition, it may give insights into the information problem in black hole physics. We have not yet explicitly mentioned it. This problem is related to the entropy interpretation problem of what and where are the degrees of freedom corresponding to the black hole entropy. We have argued that the black hole seems to hide many features inside the horizon. For instance it possibly hides all the information that the original star had before it collapsed into a black hole. But now, how can we get information to reobtain those features? If the inside of black holes are disconnected to the outside world, then classically this information seems to have disappeared. Is there an information loss, and with it a break of unitarity? It has not yet been shown that information is lost or not lost when one throws objects through a black hole, following Hawking's original suggestion of information loss. But, in this string theory description, a black hole is now dual to a lower dimensional world in which it seems information is never lost. So there is hope in solving this problem. Another place where it can be of use is in quantum field theory itself. The reason is that anti-de Sitter spacetime yields relatively easy calculations, whereas calculations on quantum fields are technically hard. For instance, one cannot yet derive the proton and neutron properties from quantum chromodynamics, the theory of quarks, a well understood theory, but extremely hard to solve. One can now try to solve these properties using the above duality.

VI. CONCLUSIONS

We have trodden a long way from the first ideas in black holes with their associated event horizons all within the context of pure classical general relativity, passed through semiclassical calculations meeting the concepts of entropy and temperature for black holes, and then through the statistical physics connection of entropy and its associated second law, arriving at the maximum number of degrees of freedom a fundamental theory, one which includes quantum gravity, can have. Surprisingly, this number goes with the area A of the region, rather than with the volume V. In turn this means first that the holographic principle should be valid, i.e., a region with boundary area A is totally described by at most $A/4$ degrees of freedom (in natural units), and second we need a fundamental theory that incorporates this principle. As we have seen, local quantum field theory is certainly not such a theory. Taking this idea seriously, one can advance that the universe can indeed be described by a model with one less dimension, in the sense, that the formulation of the fundamental theory can be done in a lower dimension. Perhaps, as string theory suggests, the fundamental theory can be formulated in the dimensions we are used to (in our case three plus one spacetime dimensions), as well as in the holographic dimensions (in our case two plus one spacetime dimensions). In this case, one should also be able to find a dictionary, or a map, between both formulations. This idea that one can trade spacetime dimensions in a fundamental description, leads one to speculate that, conceivably, spacetime itself is not a so fundamental concept. Of course, if true, such considerations are destined to enter into the philosophy dominion and radically transform our notions of what space and time are. Is the universe a hologram? Is there a shadow universe in which our bodies exist in a compressed two dimensional form? The answer lies ahead.

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[†] Electronic address: lemos@fisica.ist.utl.pt

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